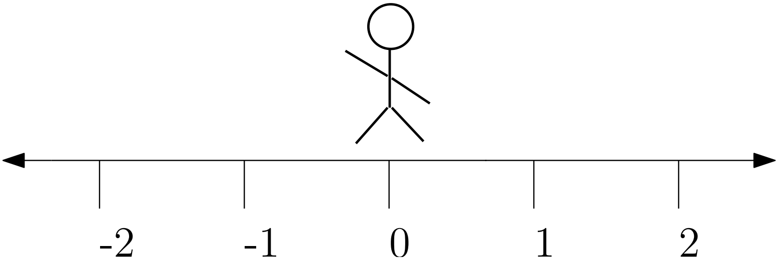
Upper and lower bounds on speed function of an excited random walk

Erin Bossen, Brian Kidd, Owen Levin, Jacob Smith, and Kevin Stangl

August 2, 2016

Simple Random Walk



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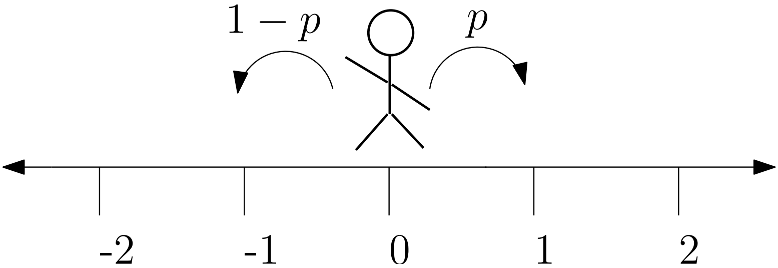

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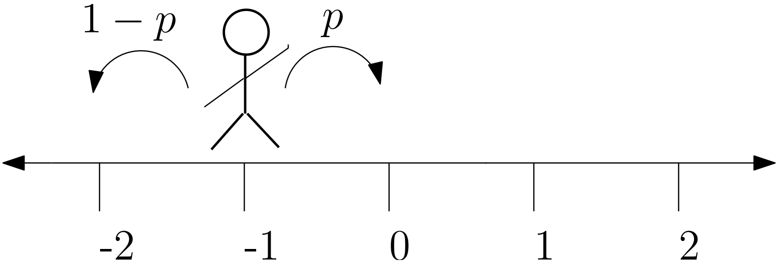

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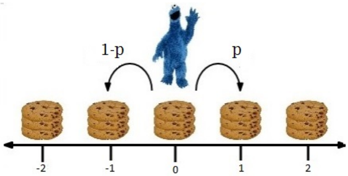
- Speed

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{n}\sum_{i=1}^{\infty} \xi_i = \mathbb{E}_0[X_1] = 2p-1 \text{ by SLLN}$$

Excited Random Walks



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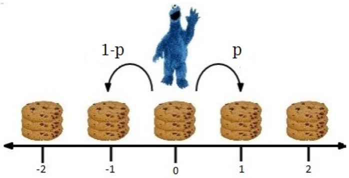

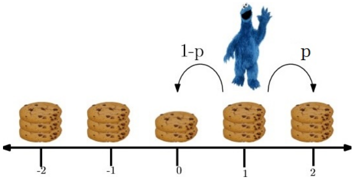

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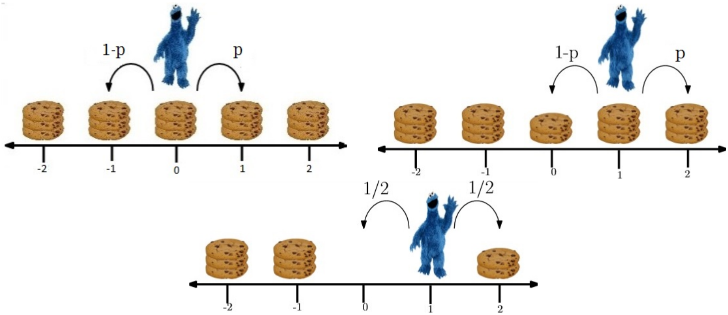

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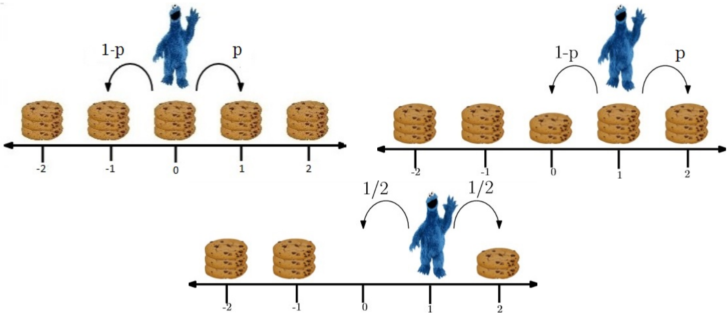

Questions

1. Transcient?
2. Recurrent?
3. Speed?

Prior Results That Structure Our Bounds

Drift defined as $\delta = M(2p - 1)$

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Theorem (Zerner '05)

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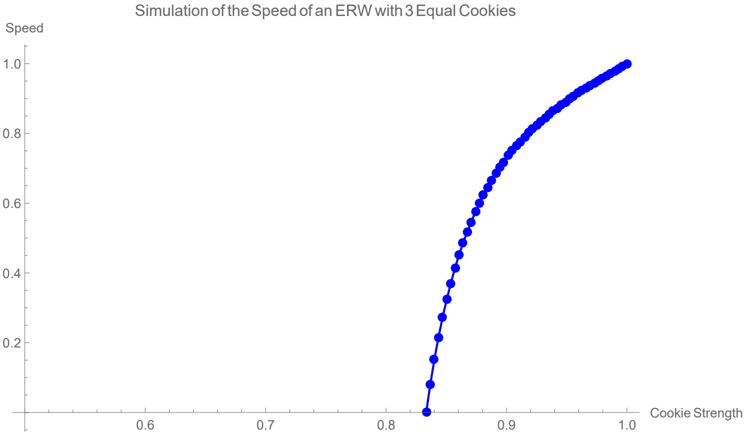
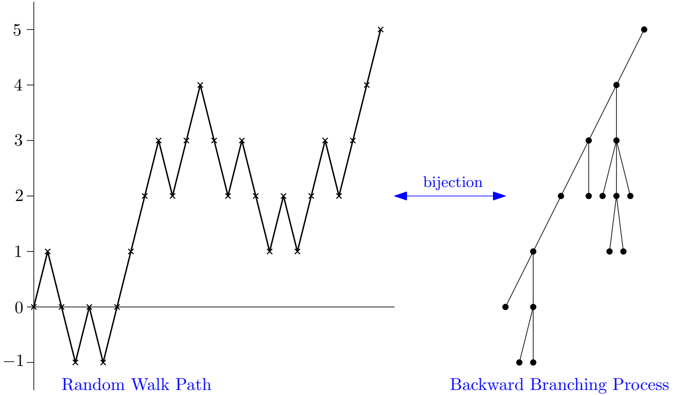
Theorem (Basdevant and Singh '08)

positive speed v $\iff \delta > 2$

Simulation of Speed



Branching Process- This is what we study!



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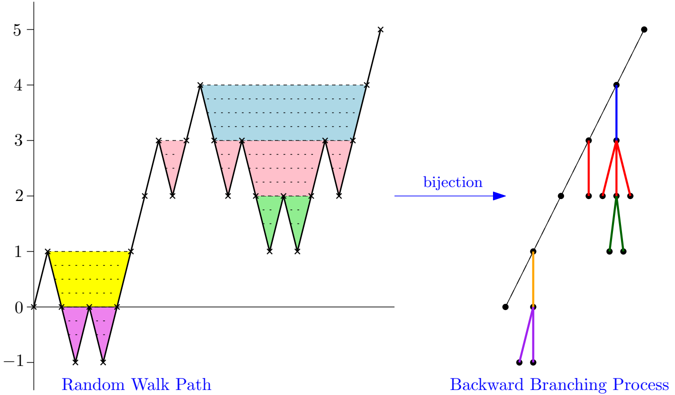

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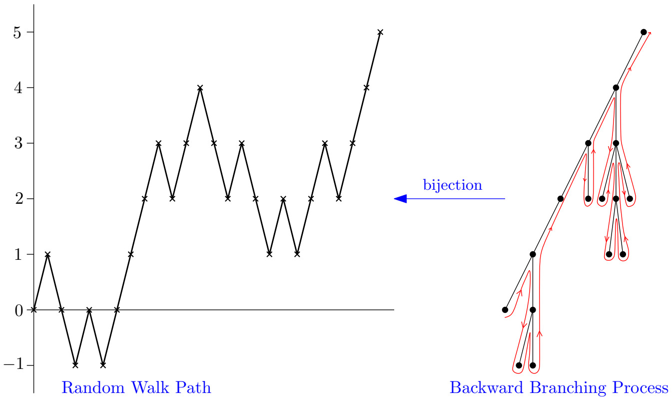

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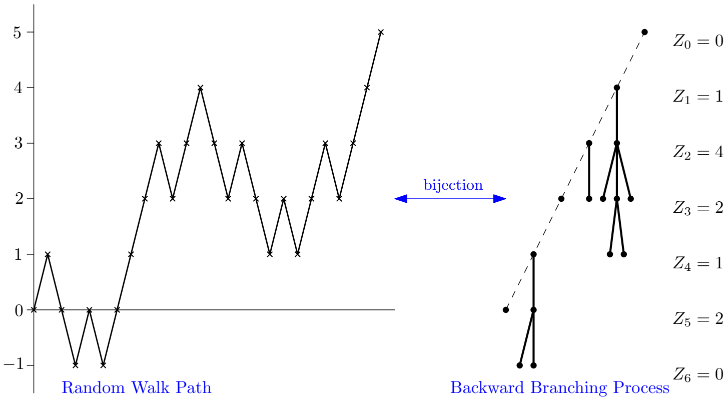

Alternate form of Speed

- $V = \lim_{n\to\infty} \frac{X_n}{n}$

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- $V = \lim_{n\to\infty} \frac{X_n}{n}$
- $\frac{1}{V} = \lim_{n\to\infty} \frac{T_n}{n}$
- $T_n = \inf\{k \geq 0 : X_k = n\}$

Hitting Time Fun!

- $T_n = n + 2($ number of total offspring of tree$)$

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- $T_n = n + 2($ number of total offspring of tree$)$
- $T_n = n + 2\sum_i Z_i$
- Now we take the limit!

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} 1 + 2\frac{1}{n}\sum_i Z_i$$

Strong Law-ish

- $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} Z_i = \mathbb{E}_\pi[Z_0] = \sum_{k\geq 0} k\pi(k)$
- Where

$$\begin{aligned}\pi(k) &= \lim_{n\to\infty} P(Z_n = k) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i\geq 0}^{n} 1_{Z_i=k}\end{aligned}$$

Looks tractable!

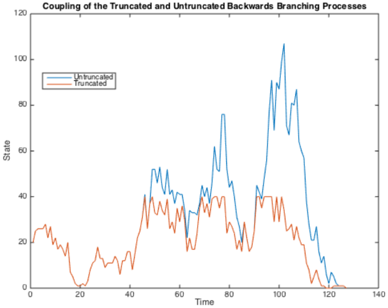
Theorem (Basdevant and Singh ‘08)

$v = \frac{1}{1+2\mathbb{E}_\pi[Z_0]}$

Truncation of Z_n

Denoted $Z_n^{(L)}$

$$P(Z_{n+1}^{(L)} = j | Z_n^{(L)} = i) = \begin{cases} P(Z_{n+1} = j | Z_n = i) & \text{if } j < L \\ P(Z_{n+1} \geq L | Z_n = i) & \text{if } j = L \end{cases}$$



Transition Matrix

- $i, j \leq 2$

$$P_{i,j} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ \end{array} & \begin{pmatrix} p & pq & pq^2 & \cdots \\ p^2 & 2p^2q & \frac{3}{2}pq^2 & \cdots \\ p^3 & \frac{3}{2}p^2q & \frac{3}{4}(pq^2 + p^2q) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

- $i > 2$ and/or $j > 2$ and $j < L$

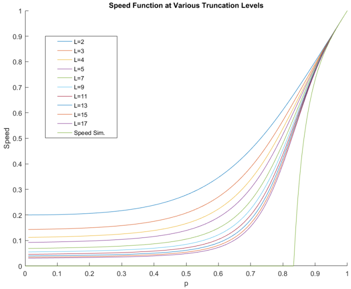
$$P_{i,j} = \frac{1}{2^{i+j-2}}\Big[\binom{i+j-3}{i-3}p^3 + \binom{i+j-3}{j-3}q^3 + 3\binom{i+j-3}{i-2}p^2q + 3\binom{i+j-3}{j-2}pq^2\Big]$$

Transition Matrix for Truncation

- $j = L$ (the last column)

$$P_{i,j} = 1 - \sum_{j=0}^{L-1} p(i,j)$$

Upper Bound Using Truncation



Theorem

$$V^{(L)}_{M,\vec{p}} \xrightarrow{L\to\infty} V_{M,\vec{p}}$$

$\mathbb{E}_k[Z_1]$ for $M = 3$ Cookies of Equal Strength

- $\mathbb{E}_k[Z_1] = \sum_{n=0}^{\infty} n \cdot \mathbb{P}(Z_1 = n | Z_0 = k) = \sum_{n=0}^{\infty} n \cdot P_{k,n}$

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- $$\begin{aligned}\mathbb{E}_0[Z_1] &= 0(p) + 1p(1-p) + 2p(1-p)^2 + (1-p)^3 \sum_{k=3}^{\infty} \frac{k}{2^{k-2}} \\ &= 4 - 9p + 7p^2 - 2p^3\end{aligned}$$

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- $\mathbb{E}_1[Z_1] = 5 - 6p - p^2 + 2p^3$
- $\mathbb{E}_k[Z_1] = k + 4 - 6p \qquad \forall k \geq M - 1$

Probability Generating Functions

- $G(s) := \sum_{k=0}^{\infty} \pi(k) s^k = \mathbb{E}_\pi[s^{Z_0}]$
- $G'(1) = \mathbb{E}_\pi[Z_0]$

Recursive Formula (Basdevant & Singh)

$$1 - G\left(\frac{1}{2-s}\right) = a(s)(1 - G(s)) + b(s)$$

- $a(s) = \frac{1}{(2-s)^{M-1}\mathbb{E}_{M-1}[s^{Z_1}]}$
- $b(s) = 1 - \frac{1}{(2-s)^{M-1}\mathbb{E}_{M-1}[s^{Z_1}]} + \sum_{k=0}^{M-2} \pi(k) \left(\frac{\mathbb{E}_k[s^{Z_1}]}{(2-s)^{M-1}\mathbb{E}_{M-1}[s^{Z_1}]} - \frac{1}{(2-s)^k}\right)$

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- $\mathbb{E}_\pi[Z_0] = \frac{b''(1)}{2(\delta-2)}$

b"(1) for $M = 3$ Cookies

$$\begin{aligned} b''(1) &= -6 + \mathbb{E}_2[Z_1] + \mathbb{E}_2[Z_1^2] \\ &\quad + \pi(0)\Big\{ -2\mathbb{E}_2[Z_1](2 + \mathbb{E}_0[Z_1]) + 6 - 2\mathbb{E}_0[Z_1] - \mathbb{E}_0[Z_1^2] + 2\mathbb{E}_0[Z_1]^2 \Big\} \\ &\quad + \pi(1)\Big\{ -2 - 2\mathbb{E}_2[Z_1](2 + \mathbb{E}_1[Z_1]) + 6 - 2\mathbb{E}_1[Z_1] - \mathbb{E}_1[Z_1^2] + 2\mathbb{E}_1[Z_1]^2 \Big\} \\ &= -2(2p-1)(12p-9) - 2(2p-1)(6p^3 - 19p^2 + 10p)\pi(0) - 2(2p-1)(-6p^3 + 4p^2)\pi(1) \end{aligned}$$

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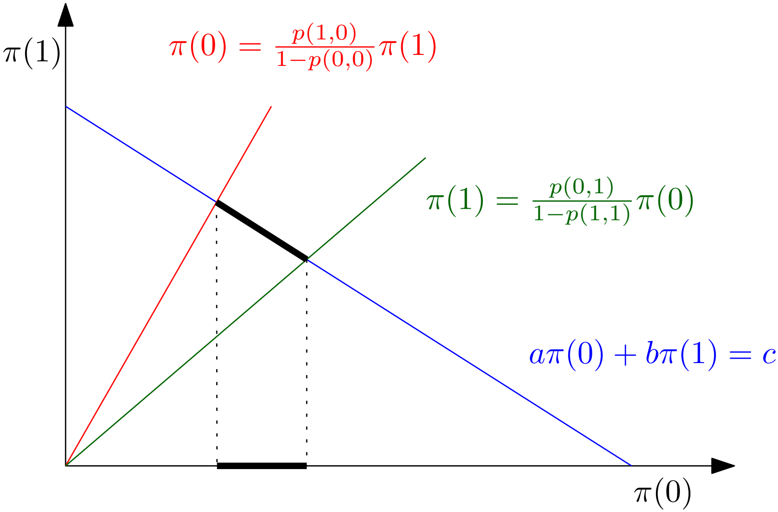
\end{aligned}
$$

$$
V = \frac{6p - 5}{6p - 5 - 2(2p-1)(12p-9) - 2(2p-1)(6p^3 - 19p^2 + 10p1)\pi(0) - 2(2p-1)(-6p^3 + 4p^2)\pi(1)}
$$

Upper and Lower Bounds



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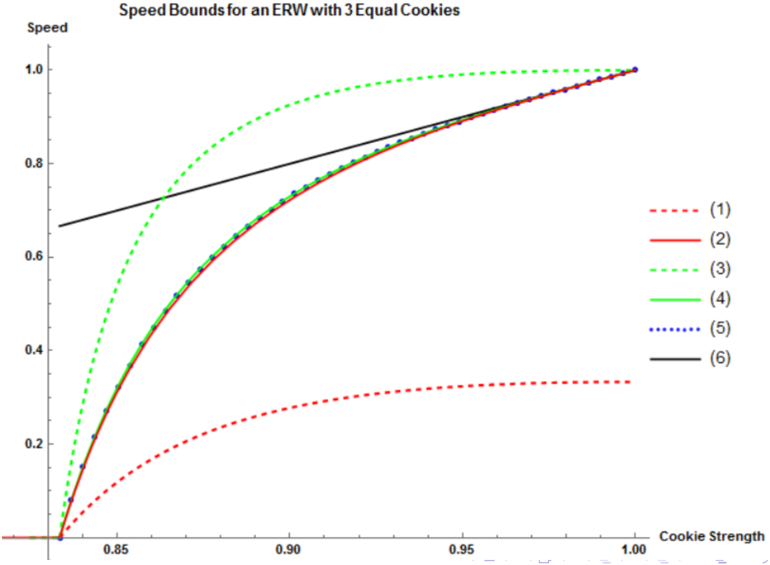

Future Research

- Find an explicit equation for the speed
- Differentiability of the speed function

Thank you for listening!

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